

APPROXIMATION ON ARCS AND DENDRITES GOING TO INFINITY IN \mathbb{C}^n (EXTENDED VERSION)

P. M. GAUTHIER E. S. ZERON (*IN MEMORIAM: HERBERT JAMES ALEXANDER 1940-1999*)

ABSTRACT. The Stone-Weierstrass approximation theorem is extended to certain unbounded sets in \mathbb{C}^n . In particular, on a locally rectifiable arc going to infinity, each continuous function can be approximated by entire functions.

AMS subject classification numbers. Primary: 32E30. Secondary: 32E25.
Key words: Tangential approximation.

1. INTRODUCTION

This work is the original version of the paper: *Approximation on arcs and dendrites going to infinite in \mathbb{C}^n* [11]. This version could not be published in its extended form because of size limitations. However, we wish to publish it because it contains a sketch of the proof of Alexander-Stolzenberg's theorem, which we announced in [11], and several lemmas on tangential approximation by polynomial and meromorphic functions which could not be included on [11]. For example, we include a not-very-known result of Arakelian in Proposition 5.

A famous theorem of Torsten Carleman [6] asserts that for each continuous function f on the real line \mathbb{R} and for each positive continuous function ϵ on \mathbb{R} , there exists an entire function g on \mathbb{C} such that

$$|f(x) - g(x)| < \epsilon(x), \text{ for all } x \in \mathbb{R}.$$

Carleman's theorem was extended to \mathbb{C}^n by Herbert Alexander [2] who replaced the line \mathbb{R} by a piecewise smooth arc going to infinity in \mathbb{C}^n and by Stephen Scheinberg [14] who replaced the real line \mathbb{R} by the real part \mathbb{R}^n of $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. In the present work, we approximate on closed subsets of area zero in \mathbb{C}^n and extend Alexander's theorem to locally rectifiable closed connected subsets $\Gamma \subset \mathbb{C}^n$ which contain no closed curves.

Let X be a subset of \mathbb{C}^n . X is a continuum if it is a compact connected set. The length and area of X are the Hausdorff 1-measure and 2-measure of X respectively. The set X is said to be of finite length at a point $x \in X$ if this point has a neighbourhood in X of finite length, and X is said to be of locally finite length if X is of finite length at each of its points. Notice that if X is a set of locally finite length, then each compact subset of X has finite length (though X itself need not be of finite length). We denote the polynomial hull of a compact set X by \hat{X} . The algebra of continuous functions defined on X is denoted by $\mathcal{C}(X)$. Finally, the definition and some properties of the first Čech cohomology group with integer coefficients $\check{H}^1(X)$ are presented in [9] and [17].

Date: February 2, 2008.

Research supported by CRSNG(Canada), FCAR(Québec) and Cinvestav(México).

2. THE ALEXANDER-STOLZENBERG THEOREM

John Wermer laid the foundations of approximation on curves in \mathbb{C}^n and prepared the way for a fundamental result of Gabriel Stolzenberg [15] concerning hulls and smooth curves (for history see [16]). In [3], Alexander comments that Stolzenberg's theorem can be improved to consider *continua of finite length* instead of *smooth curves*. We shall refer to the following version as the Alexander-Stolzenberg Theorem.

Theorem 1 (Alexander-Stolzenberg). *Let X and Y be two compact subsets of \mathbb{C}^n , with X polynomially convex and $Y \setminus X$ of zero area. Then,*

A: *Every continuous function on $X \cup Y$ which is uniformly approximable on X by polynomials is uniformly approximable on $X \cup Y$ by rational functions.*

Suppose, moreover, there exists a continuum $\Upsilon \subset \mathbb{C}^n$ such that $\Upsilon \setminus X$ has locally finite length and $Y \subset (X \cup \Upsilon)$. Then:

B: *$\widehat{X \cup Y} \setminus (X \cup Y)$ is (if non-empty) a pure one-dimensional analytic subset of $\mathbb{C}^n \setminus (X \cup Y)$.*

C: *If the map $\check{H}^1(X \cup Y) \rightarrow \check{H}^1(X)$ induced by $X \subset X \cup Y$ is injective, then $X \cup Y$ is polynomially convex.*

The proof of this theorem is implicitly contained in the papers of Stolzenberg [16] and Alexander [3], so we will devote this section to presenting a mere sketch of the proof by just indicating the necessary modifications to the existing proofs. Note also, that in this Alexander-Stolzenberg Theorem, locally finite length is required only for parts **B** and **C**.

The main arguments of the following lemma are essentially in [16, p. 188].

Lemma 1. *Let X and Y be two compact subsets of \mathbb{C}^n , with X rationally convex and $Y \setminus X$ of zero area. Then, $X \cup Y$ is rationally convex. If, moreover, X is polynomially convex, then given a point p in the complement of $X \cup Y$, there is a polynomial such that $f(p) = 0$, $0 \notin f(X \cup Y)$ and $\Re f(z) < -1$ for $z \in X$.*

Proof. The set X has a fundamental system of neighbourhoods which are rational polyhedra [15, p. 283] or [17]. Given a point p in the complement of $X \cup Y$, choose a compact rational polyhedron \tilde{X} which contains X in its interior, but $p \notin \tilde{X}$. Along with \tilde{X} , the closure K of $Y \setminus \tilde{X}$ is also rationally convex because it has zero area [9, p. 71], so there are two polynomials g and h such that $0 \notin g(K)$, $0 \notin h(\tilde{X})$ and $g(p) = h(p) = 0$.

The rational function (h/g) is smooth on K , and so $(h/g)(K)$ has zero area. Thus, we can find a complex number $\lambda \notin (h/g)(K)$ whose absolute value $|\lambda|$ is so small that the polynomial $f = h - \lambda g$ has no zeros on $\tilde{X} \cup K$. Since $X \cup Y \subset \tilde{X} \cup K$ and $f(p) = 0$, it follows that $X \cup Y$ is rationally convex.

If, in addition, X is polynomially convex, one has just to choose \tilde{X} to be a compact polynomial polyhedron (see [15] or [18, Lemma 7.4]) and the polynomial h to satisfy $\Re(h) < -1$ on \tilde{X} ; and so, for sufficiently small λ , $\Re(f) < -1$ on X . \square

To prove part **A** of Theorem 1, suppose first that the set Y is itself of zero area. Stolzenberg's proof [16, p. 187] uses the fact that the polynomial image of a finite union of smooth curves has zero area, and we also have that the image of Y under a polynomial has area zero. Now suppose we merely know that $Y \setminus X$ has zero area.

Let f be a continuous function on $X \cup Y$ which is uniformly approximable on X by polynomials and let $\epsilon > 0$. There exists a polynomial p such that $|f - p| < \epsilon/2$ on X . Since X is polynomially convex, it has a fundamental system of neighbourhoods which are polynomial polyhedra [18, Lemma 7.4]. From the continuity of $f - p$, it follows that $|f - p| < \epsilon/2$ on some polynomial polyhedron \tilde{X} containing X in its interior. Extend $p|_{\tilde{X}}$ to a continuous function \tilde{p} on $Y \setminus \tilde{X}$ so that $|f - \tilde{p}| < \epsilon/2$ on $\tilde{X} \cup Y$. Since $\tilde{X} \cup Y$ can be written as the union of \tilde{X} with a *compact* set of area zero, it follows from the first part of this proof that there is a rational function h such that $|\tilde{p} - h| < \epsilon/2$ on $\tilde{X} \cup Y$. By the triangle inequality, $|f - h| < \epsilon$ on $X \cup Y$ which concludes the proof of **A**.

Part **C** can be deduced from part **B** of Theorem 1 as presented in [16, p. 188]. No changes are required because no metrical properties are invoked. Finally, the proof of part **B** is implicitly contained in Alexander paper [3], but we need to make several remarks.

Set $\Gamma = X \cup Y$ and suppose there is a point $p \in \hat{\Gamma} \setminus \Gamma$. From Lemma 1, there is a polynomial f such that $f(p) = 0$, $0 \notin f(\Gamma)$ and $\Re(f) < -1$ on X . Fix the compact set $L = f(\Gamma)$ and the half-plane $H = \{(x, y) \in \mathbb{C} : x \geq -1/2\}$. Alexander's arguments [3] can be slightly modified to show that $\hat{\Gamma} \cap f^{-1}(\Omega)$ is a 1-dimensional analytic subset of $f^{-1}(\Omega)$, where Ω is the connected component of $\mathbb{C} \setminus L$ which contains the origin. Alexander uses the hypothesis that the set L has finite length in the whole plane \mathbb{C} . However, his argument works even if we restrict the set L to have finite length just in the half-plane H . Indeed, the intersection $L \cap H$ is the polynomial image of the compact set $\Gamma \cap f^{-1}(H)$ of finite length; recall that $\Gamma \cap f^{-1}(H) = (Y \setminus X) \cap f^{-1}(H)$ has finite length because it is compact and contained in the set $Y \setminus X$ of locally finite length. Moreover, we shall see that we just need to rewrite Lemmas 3, 5 and 6 of [3] to get the result.

For a set X , let $\#X$ denote the number ($\leq \infty$) of elements in X .

Lemma 2 (Lemma 3 of [3]). *Let Γ be a compact set in \mathbb{C}^n and f a polynomial in \mathbb{C}^n such that $\Gamma \cap f^{-1}(H)$ has finite length. For $x \in \mathbb{R}$, set $N(x) = \#\{p \in \Gamma : \Re f(p) = x\}$. Then $\int_{-1/2}^{\infty} N(x) dx < \infty$.*

Lemma 3 (Lemma 5 of [3]). *Let $L \subset \mathbb{C}$ be compact and such that $\int_{-1/2}^{\infty} N(x) dx < \infty$ where $N(x) = \#\{q \in L : \Re(q) = x\}$. Then, for every component Ω of $\mathbb{C} \setminus L$ which meets the half-plane H , there exists a finite sequence $\Omega_0, \Omega_1, \dots, \Omega_m$ of components of $\mathbb{C} \setminus L$ with $\Omega_0 =$ the unbounded component, $\Omega_m = \Omega$ and (Ω_{j-1}, Ω_j) amply adjacent through rectangles $R_j = [a, b] \times [c_{j-1}, c_j]$ contained in H for $j = 1, 2, \dots, m$.*

Lemma 2 need not be commented, and Lemma 3 holds by firstly choosing a horizontal line segment $[a, b] \times c \subset \Omega \cap H$ in the original proof.

Lemma 4 (Lemma 6 of [3]). *Let Γ be a compact subset of \mathbb{C}^n and f a polynomial in \mathbb{C}^n . Set $L = f(\Gamma) \subset \mathbb{C}$. Suppose that $\int_{-1/2}^{\infty} N(x) dx < \infty$ and that $L \cap H$ is contained in a continuum L_1 whose intersection $L_1 \cap H$ is of finite length. Let (Ω_1, Ω_2) be a pair of components of $\mathbb{C} \setminus (L \cup L_1)$ which are amply adjacent through the square $R = [a, b] \times [c_1, c_2] \subset H$. Suppose $\hat{\Gamma} \cap f^{-1}(\Omega_i)$ is a (possibly empty) pure 1-dimensional analytic subset of $f^{-1}(\Omega_i)$ for $i = 1$. Then, the same is true for $i = 2$.*

Alexander proves that $\hat{\Gamma} \cap f^{-1}(D^\circ)$ is a pure 1-dimensional analytic subset of $f^{-1}(D^\circ)$ where D° is an open set contained in $R \cap \Omega_2$. He deduces then that $\hat{\Gamma} \cap$

$f^{-1}(\Omega_2)$ is also a pure 1-dimensional analytic set in $f^{-1}(\Omega_2)$ by using Lemma 11 of [16]. This lemma is quite amazing because the component Ω_2 may not be completely contained in H . In our case, the analysis is done at a neighbourhood of the square $R \subset H$, so $\widehat{\Gamma} \cap f^{-1}(D^o)$ is analytic because L_1 has finite length in H . The result in Lemma 4 then follows, because the Stolzenberg Lemma 11 of [16], which Alexander invokes, contains no metric restrictions in its hypotheses.

We conclude the proof of part **B** of Theorem 1 following Alexander's original arguments. If the equality $X \cup \Upsilon = \Gamma = X \cup Y$ holds, we let Ω be the connected component of $\mathbb{C} \setminus L$ which contains the origin. Apply Lemmas 2 and 3 to get a sequence $\Omega_0, \Omega_1, \dots, \Omega_m = \Omega$. Finally, in Lemma 4, take $L_1 = f(\Upsilon) \subset L$; recall that $\Upsilon \cap f^{-1}(H)$ is a compact set of finite length because it is contained in $\Upsilon \setminus X$. Then, noting that $f^{-1}(\Omega_0) \cap \widehat{\Gamma} = \emptyset$, we conclude inductively that $f^{-1}(\Omega) \cap \widehat{\Gamma}$ is a 1-dimensional analytic subset of $f^{-1}(\Omega)$. Hence: $\widehat{X \cup Y} \setminus (X \cup Y)$ is analytic at an arbitrary point $p \in \widehat{X \cup Y} \setminus (X \cup Y)$.

Now suppose that $X \cup Y$ is strictly contained in $X \cup \Upsilon$. Let $p \in \widehat{\Gamma} \setminus \Gamma$ as above. Modify Υ to obtain Υ_0 such that $p \notin \Upsilon_0$ but Υ_0 is a continuum with $\Upsilon_0 \setminus X$ of finite length and $Y \subset (X \cup \Upsilon_0)$ (say by radial projection to the boundary inside a ball containing p in its interior, centered off Υ , and disjoint from Γ). By the previous paragraph, $\widehat{X \cup \Upsilon_0} \setminus (X \cup \Upsilon_0)$ is analytic. By Lemma 7 of [3], $\widehat{X \cup Y} \setminus (X \cup Y)$ is analytic at p .

An arc Υ , that is, the homeomorphic image of an interval of the real line, is of finite length at a point $x \in \Upsilon$ if and only if Υ is (locally) rectifiable at x . A direct consequence of the Alexander-Stolzenberg theorem is that every compact arc Υ which is locally rectifiable everywhere except perhaps at finitely many of its points is polynomially convex and the approximation condition $C(\Upsilon) = P(\Upsilon)$ holds; notice that Υ may be of infinite length.

It is natural to ask whether the connectivity can be dropped in these considerations. In fact, Alexander [4] gave an example of a compact set Y of finite length in \mathbb{C}^2 for which $\widehat{Y} \setminus Y$ is *not* a pure one-dimensional analytic subset of $\mathbb{C}^2 \setminus Y$. Thus, the connectivity cannot be dropped in the rectifiable Stolzenberg Theorem of Alexander. Moreover, the following example shows that we cannot finesse Theorem 1 by enclosing Y in a continuum of finite length, although it is known that one can always construct a compact arc Γ which meets every component of Y (so $Y \cup \Gamma$ is connected) and $\Gamma \setminus Y$ is *locally* rectifiable.

Example 1. *There exists a discrete bounded set in $\mathbb{C} \setminus \{0\}$ such that no continuum containing this sequence has finite length.*

Consider the set E consisting of the complex numbers $w_{j,k} = k/j^2 + \sqrt{-1}/j$, for $j = 1, 2, \dots$ and $k = 0, 1, \dots, j$. It is easy to see that E is contained in the disjoint union of the closed balls $\overline{B}_{j,k}$ with respective centers $w_{j,k}$ and radii $\frac{1}{2(j+1)^2}$. Hence, each continuum which contains E has to meet the center and the boundary of each ball $\overline{B}_{j,k}$, so its length has to be greater than $\sum_{j>1} \frac{j+1}{2(j+1)^2} = \infty$.

3. APPROXIMATION ON UNBOUNDED SETS

We now pass from approximation on compacta to approximation on closed sets. Let Y be a closed subset of \mathbb{C}^n and \mathcal{F} a subclass of $\mathcal{C}(Y)$. We say that a function

f defined on Y can be uniformly (resp. tangentially) approximated by functions in \mathcal{F} if for each positive constant ϵ (resp. positive continuous function ϵ on Y) there is a $g \in \mathcal{F}$ such that $|f - g| < \epsilon$ on Y . As \mathcal{F} we are interested in the restrictions to Y of the class $\mathcal{O}(\mathbb{C}^n)$ of entire functions and the class of meromorphic functions on \mathbb{C}^n whose singularities do not meet Y . In the latter case, we say that f can be uniformly (resp. tangentially) approximated by meromorphic functions on \mathbb{C}^n . Recall that these meromorphic functions can be expressed as a quotient p/q of entire functions p and q with $q(z) \neq 0$ for all $z \in Y$ because the second Cousin problem can be solved in \mathbb{C}^n .

If Y is compact, then of course uniform and tangential approximation are equivalent and we may replace the classes of entire and meromorphic functions on \mathbb{C}^n by the classes of polynomials and rational functions respectively.

We say that Y is a set of uniform (resp. tangential) approximation by functions in the class \mathcal{F} if each $f \in \mathcal{C}(Y)$ can be uniformly (resp. tangentially) approximated by functions in \mathcal{F} . Of course, as we have defined them, such sets Y cannot have any interior. In the literature, one also finds a more generous notion of sets of uniform or tangential approximation, which allows some sets having interior.

Before going any further, we should point out that, sets of uniform approximation and sets of tangential approximation by holomorphic functions are in fact the same. This was proved by Norair Arakelian in his doctoral dissertation [5] in \mathbb{C} . His proof works verbatim in \mathbb{C}^n . Since this fact is not well known and the proof is short we include it.

Proposition 1 (Arakelian). *Let Y be a closed subset of \mathbb{C}^n and let \mathcal{F} be either the class of functions holomorphic on Y or the class of entire functions. Then, Y is a set of uniform approximation by functions in the class \mathcal{F} if and only if it is a set of tangential approximation by functions in the same class.*

Proof. Suppose Y is a set of uniform approximation, $f \in \mathcal{C}(Y)$ and ϵ is a positive continuous function on Y . Set $\psi = \ln \epsilon$. There exists a function $g_1 \in \mathcal{F}$ such that $|\psi - g_1| < 1$ on Y . Setting $h = \exp(g_1 - 1)$, consider the functions $f/h \in \mathcal{C}(Y)$. There exists a function $g_2 \in \mathcal{F}$ such that $|f/h - g_2| < 1$ on Y . Then, $|f - hg_2| < |h| = \exp(\Re(g_1) - 1) < \exp \psi = \epsilon$. This completes the proof. \square

The following is a non-compact version of the Stone-Weierstrass Theorem.

Proposition 2. *A closed set $\Gamma \subset \mathbb{C}^n$ is a set of tangential approximation by entire functions if and only if one can approximate (in the tangential sense) the projections $\Re(z_m)$ for $m = 1, \dots, n$.*

Proof. The necessity is trivial. Moreover, if one can approximate $\Re(z_m)$, one can approximate $\Im(z_m)$ as well since $\Im(z_m) = i(\Re(z_m) - z_m)$. Let I be the natural diffeomorphism of \mathbb{C}^n onto the real part \mathbb{R}^{2n} of \mathbb{C}^{2n} . That is: $I_1(z) = \Re(z_1)$, $I_2(z) = \Im(z_1)$, $I_3(z) = \Re(z_2)$, $I_4(z) = \Im(z_2)$, etc., for $z \in \mathbb{C}^n$. Given two continuous function $f, \epsilon \in \mathcal{C}(\Gamma)$ with ϵ real positive, we may extend both of them continuously to all of \mathbb{C}^n while keeping ϵ positive. By the theorem of Scheinberg (see introduction), there is an entire function $F \in \mathcal{O}(\mathbb{C}^{2n})$ such that $|f(z) - F \circ I(z)| < \epsilon(z)/2$ for $z \in \mathbb{C}^n$.

Since F is uniformly continuous on compact subsets of \mathbb{C}^{2n} , there is a positive continuous function δ on \mathbb{C}^n such that $|F \circ I(z) - F(w)| < \epsilon(z)/2$, for each $z \in \mathbb{C}^n$ and each $w \in \mathbb{C}^{2n}$ for which $|I(z) - w| < \delta(z)$.

By hypotheses, we can approximate each I_m on Γ by entire functions and so there exists an entire mapping $h : \mathbb{C}^n \rightarrow \mathbb{C}^{2n}$ with $|I - h| < \delta$ on Γ . Thus, $|F \circ I - F \circ h| < \epsilon/2$ on Γ . By the triangle inequality, $|f - F \circ h| < \epsilon$ on Γ . The function $F \circ h$ is entire because h and F are holomorphic. \square

An interesting consequence of this result is that neither projection \Re nor \Im , in the complex plane $z \in \mathbb{C}$, can be tangentially approximated in the classical examples where the tangential approximation fails to hold, although uniform approximation may sometimes be possible.

Example 2. *Let*

$$Y = \bigcup_{j=0}^{\infty} Y_j,$$

where $Y_0 = [0, +\infty) \times \{0\}$, and for $j = 1, 2, \dots$,

$$Y_j = \left([0, j] \times \left\{ \frac{1}{2j}, \frac{1}{2j+1} \right\} \right) \cup \left(\{j\} \times \left[\frac{1}{2j}, \frac{1}{2j+1} \right] \right).$$

Then, on Y both \Re and \Im can be approximated uniformly but not tangentially by entire functions.

Proof. In his doctoral thesis, Arakelian [5] gave a complete characterization for sets of uniform approximation, from which it follows that Y is not a set of uniform approximation and *a fortiori* not a set of tangential approximation. Thus, by Proposition 2, \Re and \Im cannot be approximated tangentially. We show that they can be approximated uniformly.

Fix $\epsilon > 0$ and set $Z_\epsilon = \{z : |\Im(z)| \leq \epsilon\}$ and $W_\epsilon = Y \setminus Z_\epsilon$. We may assume Z_ϵ and W_ϵ disjoint (by choosing an appropriate smaller ϵ if necessary). Now, define the function

$$f = \begin{cases} \epsilon & \text{on } Z_\epsilon \\ \Im & \text{on } W_\epsilon \end{cases}.$$

Invoking again Arakelian's work (see [5], [10, p.245] or [8]), we deduce the existence of an entire function g such that $|f - g| < \epsilon$ on $Z_\epsilon \cup W_\epsilon$. Hence, $|\Im - g| < 2\epsilon$ on Y , so \Im and $\Re(z) = z - i\Im(z)$ can both be approximated uniformly on Y by entire functions. \square

It is interesting to compare Propositions 1 and 2 in the light of the previous example.

We should also notice that in Proposition 2 we can ask that the approximating functions be holomorphic merely in a neighbourhood of Γ . We thus have that each continuous function $f \in \mathcal{C}(\Gamma)$ can be approximated (in the tangential sense) by functions holomorphic in a neighbourhood of Γ if and only if every projection $\Re(z_m)$ can. This result suggests the following:

Proposition 3. *Every closed set $\Gamma \subset \mathbb{C}^n$ of area zero is a set of tangential approximation by meromorphic functions in \mathbb{C}^n .*

Proof. Let $f, \epsilon \in \mathcal{C}(\Gamma)$ be two continuous functions with ϵ real and positive, we must construct a meromorphic function F such that $|F(z) - f(z)| < \epsilon(z)$ on Γ . Let B_0 be the empty set and \overline{B}_k closed balls of radius k and center in the origin.

Lemma 5. *Each continuous function $h \in \mathcal{C}(\overline{B}_k \cup \Gamma)$ which can be uniformly approximated by polynomials in \overline{B}_k can be uniformly approximated on $D = \overline{B}_k \cup (\Gamma \cap \overline{B}_{k+1})$ by rational functions whose singularities do not meet Γ .*

Proof. From Theorem 1.A, there exists a rational function a/b such that $|(a/b)(z) - h(z)| < \delta$ for $z \in D$ and $0 \notin b(D)$. Notice that $b(\Gamma)$ has zero area, so we may choose a complex number $\lambda \notin b(\Gamma)$ with absolute value so small such that $\lambda \notin b(D)$ and $\left| \frac{a(z)}{b(z) - \lambda} - h(z) \right| < \delta$ for $z \in D$. \square

The proof of the proposition now follows a classical inductive process. There exists a rational function F_1 whose singularities do not meet Γ and such that $|F_1(z) - f(z)| < (\frac{2}{3} - 2^{-1})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_1$ by the previous lemma. Proceeding by induction, we shall construct a sequence of rational functions F_k which converges uniformly on compact sets to a meromorphic function with the desired properties.

Given a rational function F_k whose singularities do not meet Γ and such that $|F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ in $\Gamma \cap \overline{B}_k$, let h_k be a continuous function identically equal to zero on \overline{B}_k and such that $|h_k(z) + F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_{k+1}$ as well. Fix a real number $0 < \lambda_k < 1$ strictly less than $\epsilon(z)$ for every $z \in \Gamma \cap \overline{B}_{k+1}$.

Applying Lemma 5, there exists a rational function R_k whose singularities do not meet $\overline{B}_k \cup \Gamma$ and such that $|R_k(z) - h_k(z)| < 2^{-1-k}\lambda_k$ for $z \in \overline{B}_k \cup (\Gamma \cap \overline{B}_{k+1})$. Thus, the singularities of the rational function $F_{k+1}(z) = F_k(z) + R_k(z)$ do not meet Γ and $|F_{k+1}(z) - f(z)| < (\frac{2}{3} - 2^{-1-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_{k+1}$ by the triangle inequality.

Notice that $F_{k+1}(z) - F_k(z)$ is holomorphic and its absolute value is less than 2^{-1-k} inside \overline{B}_k , so the sequence F_k converges to a meromorphic function with the desired properties. \square

Similar inductive processes were originally employed to prove Carleman's theorem, stated in the introduction, which asserts that the real line \mathbb{R} in \mathbb{C} is a set of tangential approximation by entire functions. Alexander [2] extended Carleman's theorem to piecewise smooth arcs Γ going to infinity in \mathbb{C}^n . That is, Γ is the image of the real axis under a proper continuous embedding (a curve without self-intersections, *going to infinity in both directions*). We should mention that this problem had been considered independently by Bernard Aupetit and Lee Stout (see Aupetit's book [1]). As a consequence of the Alexander-Stolzenberg Theorem, we also have the following further extension of Carleman's theorem, which was conjectured by Aupetit in [1] and announced by Alexander in [2].

Proposition 4. *Let Γ be an arc which is locally rectifiable everywhere, except perhaps in a discrete subset, and going to infinity in \mathbb{C}^n . Besides, let ϵ be a strictly positive continuous function on Γ . Then, for each $f \in \mathcal{C}(\Gamma)$, there exists an entire function g on \mathbb{C}^n such that $|f(z) - g(z)| < \epsilon(z)$, for all $z \in \Gamma$. That is, Γ is a set of tangential approximation by entire functions.*

Alexander's proof (see also [1]), for the case that Γ is smooth, relies ingeniously on the topology of arcs and the original Stolzenberg Theorem for smooth curves. It works also when the arc Γ is locally rectifiable everywhere except perhaps in a discrete subset. One only needs to rewrite Lemma 1 of [2], using the following corollary of Theorem 1.

Corollary 1. *Let X and Y be two compact subsets of \mathbb{C}^n such that X is polynomially convex, Y is connected and $Y \setminus X$ is locally of finite length everywhere except perhaps at finitely many of its points. If the map $\tilde{H}^1(X \cup Y) \rightarrow \tilde{H}^1(X)$ induced by $X \subset X \cup Y$ is injective, then $X \cup Y$ is polynomially convex and every continuous function $f \in \mathcal{C}(X \cup Y)$ which can be approximated by polynomials in X can be approximated by polynomials on the union $X \cup Y$.*

Proof. Let $\{y_j\}$ be the points where $Y \setminus X$ is not of finite length. It is easy to see that $X \cup \{y_j\}$ is polynomially convex and f can be approximated by polynomials in $X \cup \{y_j\}$, so the result follows from Theorem 1, Lemma 1 and the Oka-Weil theorem. \square

We can also approximate by entire functions on unbounded sets which are more general than arcs, but first, we need to introduce the polynomially convex hull of non-compact sets:

Definition. Given an arbitrary subset Y of \mathbb{C}^n , its polynomially convex hull is defined by $\hat{Y} = \bigcup \left\{ \hat{K} : K \subset Y \text{ is compact} \right\}$.

Proposition 5. *Let Γ be a closed set in \mathbb{C}^n of zero area such that $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded for every compact set $D \subset \mathbb{C}^n$. Let B_1 be an open ball with center in the origin which contains the closure of $\hat{\Gamma} \setminus \Gamma$. That is, the set $B_1 \cup \Gamma$ contains the hull \hat{K} of every compact set $K \subset \Gamma$.*

Then, given two continuous functions $f, \epsilon \in \mathcal{C}(\Gamma)$ such that ϵ is real positive and f can be uniformly approximated by polynomials on $\Gamma \cap \overline{B_1}$, there exists an entire function F such that $|F(z) - f(z)| < \epsilon(z)$ for $z \in \Gamma$.

Proof. Let B_0 be the empty set, B_1 as in the hypotheses and B_k open balls with center in the origin such that each B_k contains the closure of $\widehat{\Gamma \cup \overline{B_{k-1}}} \setminus \Gamma$. That is, the set $B_k \cup \Gamma$ contains the hull \hat{K} of every compact set $K \subset (\Gamma \cup \overline{B_{k-1}})$. Define X_k to be the polynomially convex hull of $\overline{B_{k+1}} \cap (\Gamma \cup \overline{B_{k-1}})$, so $X_k \subset (B_k \cup \Gamma)$. The compact sets X_k and $X_k \cap \overline{B_k}$ are both polynomially convex.

The given hypotheses automatically imply that there exists a polynomial F_1 such that $|F_1(z) - f(z)| < (\frac{2}{3} - 2^{-1})\epsilon(z)$ on $\Gamma \cap \overline{B_1}$. Proceeding by induction, we shall construct a sequence of polynomials F_k which converges uniformly on compact sets to an entire function with the desired properties.

Given a polynomial F_k such that $|F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ on $\Gamma \cap \overline{B_k}$, let h_k be a continuous function equal to F_k on $\overline{B_k}$ and such that $|h_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B_{k+1}}$ as well. Fix a real number $0 < \lambda_k < 1$ strictly less than $\epsilon(z)$ for every $z \in \Gamma \cap \overline{B_{k+1}}$.

Notice that $X_k = (X_k \cap \overline{B_k}) \cup (\Gamma \cap \overline{B_{k+1}})$. Hence, by Theorem 1.A, the function h_k can be approximated by rational functions on X_k because $X_k \cap \overline{B_k}$ is polynomially convex and Γ has zero area. Moreover, the functions h_k can be approximated by polynomials by the Oka-Weil theorem. Thus, there exists a polynomial F_{k+1} such that $|F_{k+1}(z) - h_k(z)| < 2^{-1-k}\lambda_k$ for $z \in X_k$, and so $|F_{k+1}(z) - f(z)| < (\frac{2}{3} - 2^{-1-k})\epsilon(z)$ on $\Gamma \cap \overline{B_{k+1}}$.

Finally, the inequality $|F_{k+1}(z) - F_k(z)| < 2^{-1-k}$ holds for $z \in \overline{B_{k-1}}$, so the sequence F_k converges to an entire function with the desired properties. \square

On the other hand, if the equality $\widehat{\Gamma} = \Gamma$ holds as well in the last proposition, we can choose the empty set instead of the open ball B_1 (because the proof is an inductive process); and so Γ becomes a set of tangential approximation by entire functions. There are many closed sets Γ which satisfy the hypotheses of the last proposition. For example, we have the following.

Theorem 2. *Let Γ be closed connected set of locally finite length in \mathbb{C}^n whose first cohomology group $\check{H}^1(\Gamma)$ vanishes (Γ contains no simple closed curves). Then, Γ is a set of tangential approximation by entire functions.*

Proof. The proof strongly uses the topology of Γ . We show that each point of Γ has finite order, that is, has a basis of neighbourhoods in Γ having finite boundaries. Given a point $z \in \Gamma$, let B_r be the open ball in \mathbb{C}^n of radius r and center z . Since Γ is locally of finite length, the intersection of Γ with the closed ball \overline{B}_r has finite length, so the intersection of Γ with the boundary of B_s must be a finite set for almost all radii $0 < s < r$. Whence, each sub-continuum of Γ is locally connected [13, p. 283]. On the other hand, there are no simple closed curves contained in Γ because $\check{H}^1(\Gamma) = 0$, so each sub-continuum of Γ is a dendrite, that is, a locally connected continuum containing no simple closed curves. In particular, if Γ is compact, then it is a dendrite.

Notice the following lemma.

Lemma 6. *Each compact subset $K \subset \Gamma$ is contained in a sub-continuum (dendrite) of Γ .*

Proof. Since Γ is locally connected, the set K is contained in a finite union of sub-continua of Γ . The lemma now follows since Γ is arcwise connected (see Theorem 3.17 of [12]). \square

Let D be a compact set in \mathbb{C}^n . Notice that $D \cup \Gamma$ may contain simple closed curves Υ with $D \cap \Upsilon \neq \emptyset$ but $\Upsilon \not\subset D$. We shall call such a simple closed curve $\Upsilon \subset (D \cup \Gamma)$ a *loop*. We show there exists a ball which contains all of these loops. Henceforth, let B_r be open balls of radii r and center in the origin, and choose a radius $s > 0$ such that $D \subset B_s$. Recall that $\Gamma \cap \overline{B}_{s+1}$ has finite length, so there exists a ball B_t with $s < t < s + 1$ such that Γ meets the boundary of B_t only in a finite number of points $Q = \{q_1, \dots, q_m\}$. Let $\{\Upsilon_j\}$ be the possible loops which meet the complement of B_t . The set $\bigcup \{\Upsilon_j\} \setminus B_t$ is contained in Γ and can be expressed as the union of compact arcs (not necessarily disjoint) which lie outside of \overline{B}_t except for their two end points which lie in Q . Since Γ cannot contain simple closed curves, two different arcs cannot share the same end points, and there can only be finitely many such arcs. Hence, there exists a ball B_δ which contains all the loops Υ , and $D \subset B_\delta$.

We shall show that $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded. Without loss of generality, we may suppose that D is a closed ball. Since Γ is connected, the hull $\widehat{D \cup \Gamma}$ is equal to $\bigcup_{r \geq \delta} \widehat{K}_r$, where \widehat{K}_r is the connected component of $\overline{B}_r \cap (D \cup \Gamma)$ which contains D . We can prove that $\widehat{K}_r = \widehat{K}_\delta \cup K_r$, for every $r \geq \delta$, using Alexander's original argument. The following lemma is a literal translation of Lemma 1.(a) of [2], to our context.

Lemma 7. *For every $r \geq \delta$, $\widehat{K}_r = \widehat{K}_\delta \cup \tau_r$ where $\tau_r = \overline{K_r \setminus K_\delta}$.*

Since the notation is quite complicated and different from Alexander's, and we need to invoke Theorem 1.B, we shall include the proof of Lemma 7, but first we conclude the proof of the theorem.

By Lemma 7, the set $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded because $\widehat{K}_r = \widehat{K}_\delta \cup \tau_r = \widehat{K}_\delta \cup K_r$ and $\widehat{D \cup \Gamma} = (\widehat{K}_\delta \cup \Gamma)$. Moreover, the equality $\widehat{\Gamma} = \Gamma$ holds as well because each compact subset of Γ is contained in a dendrite of finite length and is polynomially convex (see Lemma 6 and Alexander's work [3]), so we can deduce from Proposition 5 that Γ is a set of tangential approximation. \square

Proof of Lemma 7. Let $T_r = \widehat{K}_\delta \cup \tau_r$ be the set on the right hand side of the asserted equality. Clearly, we have $T_r \subset \widehat{K}_r \subset \widehat{T}_r$ (the second inclusion is in fact equality). Thus it suffices to show that T_r is polynomially convex. Arguing by contradiction, we suppose otherwise. By Theorem 1.B, $\widehat{T}_r \setminus T_r$ is a 1-dimensional analytic subvariety of $\mathbb{C}^n \setminus T_r$.

Let V be a non-empty irreducible analytic component of $\widehat{T}_r \setminus T_r$. We claim that $\overline{V} \setminus K_r$ is an analytic subvariety of $\mathbb{C}^n \setminus K_r$. Since $T_r = \widehat{K}_\delta \cup \tau_r$, it suffices to verify this locally at a point $x \in \overline{V} \cap Q$ where

$$Q = \widehat{K}_\delta \setminus K_\delta.$$

By Theorem 1.B, both \widehat{K}_r and Q are analytic near x , where *near* x refers to the intersection of sets with *small enough* neighbourhoods of x , here and below. Furthermore, near x , $\overline{V} \subset \widehat{K}_r$, $V \subset \widehat{K}_r \setminus Q$ and $Q \subset \widehat{K}_r$. Thus, near x , Q is a union of some analytic components of \widehat{K}_r . It follows that near x , \overline{V} is just a union of some of the other local analytic components of \widehat{K}_r at x ; in fact, near x , $\overline{V} = V \cup \{x\}$. Put

$$W = \overline{V} \setminus K_r.$$

Then W is an irreducible analytic subset of $\mathbb{C}^n \setminus K_r$ and moreover,

$$\overline{W} \setminus W \subset K_\delta \cup \tau_r = K_r.$$

Thus $\overline{W} \subset \widehat{K}_r$ by the maximum principle.

Fix a point $p \in V \subset W$. Since $p \notin T_r$, we have $p \notin \widehat{K}_\delta$ and therefore there exists a polynomial h such that $h(p) = 0$ and $\Re h < 0$ on \widehat{K}_δ . By the open mapping theorem, either $h(W)$ is an open neighbourhood of 0 or $h \equiv 0$ on W . In the latter case, $h \equiv 0$ on \overline{W} and so $\overline{W} \setminus W$ is disjoint from K_δ . This implies that $\overline{W} \setminus W \subset \tau_r$ so $W \subset \tau_r$. We have a contradiction because τ_r is contained in a dendrite of finite length and is polynomially convex (see Lemma 6 and Alexander's work [3]), and moreover, a dendrite cannot contain a 1-dimensional analytic set. Hence, the former case holds. As $h(\tau_r)$ is nowhere dense in the plane (recall that it is of finite length), there is a small complex number $\alpha \in h(W)$ such that $\alpha \notin h(\tau_r)$. Now put $g = h - \alpha$. If α is sufficiently small, we conclude that (i) $\Re g < 0$ on \widehat{K}_δ , (ii) $g(q) = 0$ for some $q \in W$ and (iii) $0 \notin g(\tau_r)$.

Now (i) implies that the polynomial g has a continuous logarithm on \widehat{K}_δ and so, by restriction, on K_δ . We can extend this logarithm of g on K_δ to a continuous logarithm of g on K_r because of (iii), since the ball B_δ was chosen such that every simple closed curve (loop) $\Upsilon \subset K_r$ is contained in B_δ and hence in K_δ . But K_r contains $\overline{W} \setminus W$. Applying the argument principle [15, p. 271] to g on the analytic set W gives a contradiction to (ii). \square

We remark that the condition of having zero area is essential in Propositions 3 and 5, as the following example (inspired by [7]) shows.

Example 3. Let \mathcal{I} be the closed unit interval $[0, 1]$ of the real line and $K \subset \mathcal{I}$ the compact set $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. It is easy to see that the $(2 + \epsilon)$ -dimensional Hausdorff measure of the closed connected set $Y = (\mathcal{I} \times \{0\}) \cup (K \times \mathbb{C})$ in \mathbb{C}^2 is equal to zero for every $\epsilon > 0$, moreover, the equality $\widehat{Y} = Y$ holds. However, the following continuous function $f \in \mathcal{C}(Y)$ cannot be uniformly approximated by holomorphic functions in $\mathcal{O}(Y)$:

$$f(w, z) = \begin{cases} z & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose there exists a real number $\epsilon > 0$ and a holomorphic function $g \in \mathcal{O}(Y)$ such that $|f - g| < \epsilon$ on Y . We automatically have that $g(w, z)$ is bounded, analytic and constant on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \dots$. Hence, the holomorphic function $\frac{\partial g}{\partial z}$ vanishes on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \dots$ as well. Since the zero set of $\frac{\partial g}{\partial z}$ is an analytic set, this derivative must be zero in a neighbourhood of $\{0\} \times \mathbb{C}$ and hence on the connected set Y . The last statement is a contradiction to the fact that $|g(1, z) - z| < \epsilon$ for every $z \in \mathbb{C}$.

On the other hand, to see that $\widehat{Y} = Y$, notice that $Y = \bigcup_{r>0} Y_r$, where $Y_r = (\mathcal{I} \times \{0\}) \cup (K \times \Delta_r)$ and $\Delta_r \subset \mathbb{C}$ are closed discs of radius r . The set $K \times \Delta_r$ is polynomially convex because it is the Cartesian product of two polynomially convex sets in \mathbb{C} ; and so Y_r is polynomially convex because of Theorem 1.

Although connectivity, as we have emphasized, plays a crucial role in this paper, similar results can be obtained for sets whose connected components form a locally finite family. Finally, we remark that, on a Stein manifold, analogous results also hold by simply embedding the Stein manifold into some \mathbb{C}^n . A possible exception is Proposition 2, since $\Re(p)$ is not well-defined on a manifold.

REFERENCES

1. B. Aupetit, L'approximation entière sur les arcs allant à l'infini dans \mathbb{C}^n . Complex approximation (Proc. Conf., Québec, 1978), pp. 93–102, Progr. Math., 4, Birkhäuser, Boston - Basel, Mass., 1980.
2. H. Alexander, A Carleman theorem for curves in \mathbb{C}^n , *Math. Scand.* **45** (1979), no. 1, 70–76.
3. H. Alexander, Polynomial approximation and hulls in sets of finite linear measure in \mathbb{C}^n , *Amer. J. Math.* **93** (1971), 65–74.
4. H. Alexander, The polynomial hull of a set of finite linear measure in \mathbb{C}^n , *J. Analyse Math.* **47** (1986), 238–242.
5. N. U. Arakelian, Certain questions of approximation theory and the theory of entire functions. (Russian) Doctoral Dissertation. Mat. Inst. Steklov., Moscow, 1970.
6. T. Carleman, Sur un théorème de Weierstrass, *Ark. för Math. Astr. Fys.* **20** (1927), 1–5.
7. S. Chacrone, P. M. Gauthier and A. Nersessian, Carleman approximation on products of Riemann surfaces, *Complex Variables Theory Appl.* **37** (1998), no. 1–4, 97–111.
8. D. Gaier, Lectures on complex approximation. Translated from the German by R. McLaughlin. Birkhäuser Boston, Inc., Boston, Mass., 1987.
9. T. W. Gamelin, Uniform algebras. Prentice-Hall, Englewood Cliffs N.J., 1969.
10. P. M. Gauthier and G. Sabidussi, Complex potential theory. Proceedings of the NATO Advanced Study Institute and the Séminaire de Mathématiques Supérieures held in Montreal, Quebec, July 26–August 6, 1993. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 439. Kluwer Academic Publishers Group, Dordrecht, 1994.
11. P. M. Gauthier and E. S. Zeron, Approximation on arcs and dendrites going to infinite in \mathbb{C}^n . *Can. Math. Bull.* **45** (2002), No. 1, pp. 80–85.
12. J. G. Hocking and G. S. Young, Topology. Dover Publications, New York, 1988.
13. K. Kuratowski, Topology Vol. II. Academic Press, New York and London, 1968.
14. S. Scheinberg, Uniform approximation by entire functions. *J. Analyse Math.* **29** (1976), 16–18.
15. G. Stolzenberg, Polynomially and rationally convex sets. *Acta Math.* **109** (1963), 259–289.
16. G. Stolzenberg, Uniform approximation on smooth curves, *Acta Math.* **115** (1966), 185–198.
17. E. L. Stout, The theory of uniform algebras. Bogden & Quigley, Tarrytown-on-Hudson NY., 1971.
18. J. Wermer, Banach algebras and several complex variables. Graduate Texts in Mathematics No. 35, Springer-Verlag, New York-Heidelberg, 1976.

Addresses

Département de mathématiques et de statistique et
 Centre de recherches mathématiques, Université de Montréal
 Université de Montréal, CP 6128 Centre Ville,
 Montréal, H3C 3J7, Canada
 e-mail gauthier@ere.umontreal.ca

Departamento de Matemáticas, Cinvestav I.P.N.
 Apartado Postal 14-740, México D.F. 07000, México.
 e-mail eszeron@math.cinvestav.mx